

On a Class of Complete and Projectively Flat Finsler Metrics

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Abstract

An (α, β) -manifold (M, F) is a Finsler manifold with the Finsler metric F being defined by a Riemannian metric α and 1-form β on the manifold M . In this paper, we classify n -dimensional (α, β) -manifolds (non-Randers type) which are positively complete and locally projectively flat. We show that the non-trivial class is that M is homeomorphic to the n -sphere S^n and (S^n, F) is projectively related to a standard spherical Riemannian manifold, and then we obtain some special geometric properties on the geodesics and scalar flag curvature of F on S^n , especially when F is a metric of general square type.

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1 Introduction

In Finsler geometry, the flag curvature is a natural extension of the sectional curvature in Riemannian geometry. For a Finsler metric on an manifold M , the flag curvature $K = K(P, y)$ usually depends on a section (flag) $P \subset T_x M$ and a vector (pole) $y \in P$. It is said to be of scalar flag curvature if $K = K(x, y)$ is a scalar function of $y \in T_x M$, and of constant flag curvature if $K = \text{constant}$. In dimension $n \geq 3$, a Riemannian metric is of scalar flag curvature if and only if it is of constant sectional curvature. For a Finsler manifold (M, F) , geodesics are locally minimizing curves on the manifold M as to the distance defined by the metric F . Geodesics are the solutions of an ODE of second order. For a point $x \in M$ and a vector $y \in T_x M$, there is a unique geodesic $\gamma = \gamma(t)$ satisfying $\gamma(0) = x, \gamma'(0) = y$, where t is the arc-length parameter of γ . Flag curvatures and geodesics are closely related through the second-variation of arc-length for geodesics, and both of them play an important role in the studies of Finsler geometry.

It is the Hilbert's Fourth Problem to study locally projectively flat metrics. A Finsler metric is called locally projectively flat if its geodesics are locally straight. The Beltrami Theorem states that a Riemannian metric is locally projectively flat if and only if it is of constant sectional curvature. For locally projectively flat Finsler metrics, they are generally not of constant flag curvature, but they are always of scalar flag curvature. In [7], Z. Shen classifies locally projectively flat Finsler metrics of constant flag curvature by using the model of Funk metric. A Funk metric, defined on a strongly convex domain $\Omega \in \mathbf{R}^n$, is projectively flat with negative constant flag curvature, and particularly, it is positively complete ([5]). When Ω is the unit ball B^n , the Funk metric becomes a Randers metric. In [12], it gives a class of square metrics which are positively complete, locally projectively flat and of constant flag curvature on the unit ball B^n . In this paper, we will classify a class of (positively) complete Finsler metrics which are locally projectively flat and show some geometrical properties of their geodesics and scalar flag curvatures.

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All Finsler metrics on a manifold M in this paper are assumed to be regular, namely, they are positively definite and defined on the whole slit tangent bundle $TM - 0$. An (α, β) -metric F is a Finsler metric defined by a Riemannian metric $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and a 1-form $\beta = b_i(x)y^i$ on a manifold M , which can be expressed in the following form:

$$F = \alpha\phi(s), \quad s = \beta/\alpha,$$

where $\phi(s)$ is a function defined on an open interval $(-b_o, b_o)$ (see the regular condition of F in Section 2). For a pair of α and β , define $b := \|\beta\|_\alpha$. If $\phi(s) = 1 + s$, then $F = \alpha + \beta$ with $b < 1$ is called a Randers metric, which has a lot of special properties. Two (α, β) -metrics F and \tilde{F} are called of the same metric type if

$$F = \alpha\phi(\beta/\alpha), \quad \tilde{F} = \tilde{\alpha}\phi(\tilde{\beta}/\tilde{\alpha}) : \quad \tilde{\alpha} = \sqrt{\alpha^2 + \epsilon\beta^2}, \quad \tilde{\beta} = k\beta,$$

where ϵ, k are constant. In this paper, we mainly consider the (α, β) -metric $F = \alpha\phi(\beta/\alpha)$ with $\phi(s)$ being defined by

$$\{1 + (k_1 + k_3)s^2 + k_2s^4\}\phi''(s) = (k_1 + k_2s^2)\{\phi(s) - s\phi'(s)\}, \quad (\phi(0) = 1, \quad k_2 \neq k_1k_3), \quad (1)$$

where k_1, k_2, k_3 are constant. If $k_2 = k_1k_3$, then F is of Randers metric type. A special metric type of (1), called a general square metric type, is defined by

$$F = F_\epsilon^\pm := \alpha + \epsilon\beta \pm \frac{\beta^2}{\alpha}, \quad (\epsilon = \text{constant}), \quad (2)$$

in which F_2^+ is called a square metric in some literatures ([2] [11] [12]), and the metric F_0^\pm also has some special properties ([13]). Note that F in (1) is of the metric type F_0^\pm if and only if $k_2 = (2k_1 + 3k_3)(3k_1 + 2k_3)/25$ ([13]). It is shown in [6] that, in dimension $n \geq 3$, if an (α, β) -metric F (non Randers type, β not parallel) is locally projectively flat, then $\phi(s)$ satisfies (1) and β must be closed; while in dimension $n = 2$, there is one more class with F being of the metric type F_0^\pm and β generally being not closed ([13]). We show that in dimension $n \geq 3$, the metric F in (2) is locally projectively flat iff. F is of scalar flag curvature ([11] [15]).

Theorem 1.1 *Let (M, F) be an $n(\geq 2)$ -dimensional Finsler manifold, where $F = \alpha\phi(\beta/\alpha)$ is an (α, β) -metric on M . Assume that (C1) F is of non-Randers metric type; (C2) $n \geq 3$ if F is of the metric type F_0^\pm in (2); (C3) (M, F) is positively complete; (C4) F is locally projectively flat; (C5) $b(x_0) = \sup_{x \in M} b(x)$ at some point $x_0 \in M$. Then one of the following cases holds:*

- (i) $F = \alpha$ is a complete Riemannian metric of constant sectional curvature.
- (ii) F is flat-parallel (α is flat and β is parallel with respect to α).
- (iii) M is homeomorphic to the n -sphere S^n in the $(n+1)$ -dimensional Euclidean space, and (S^n, F) is projectively related to a Riemannian metric on S^n of positive constant sectional curvature. So all geodesics of (S^n, F) are closed.

Further, if F satisfies the conditions (C3)–(C5), and (C6) F is of constant flag curvature, then only cases (i) and (ii) occur.

The condition (C5) can be replaced by a weaker condition (see Remark 3.7 below). On compact manifold (without boundary), (C3) and (C5) are obviously satisfied. Without the condition (C5), there are Randers metrics or square metrics which satisfy (C3), (C4) and (C6) but do not belong to the cases (i), (ii) or (iii) of Theorem 1.1 ([8] [12]). Cases (i) and (ii) of Theorem 1.1 are trivial. So below we further study case (iii) of Theorem 1.1, and in this case, the metric F is determined by the ODE (1).

Corresponding to the ODE (1), let $u = u(t), v = v(t), w = w(t)$ satisfy the following ODEs:

$$u' = \frac{v - k_1 u}{1 + (k_1 + k_3)t + k_2 t^2}, \quad (3)$$

$$v' = \frac{u(k_2 u - k_3 v - 2k_1 v) + 2v^2}{u[1 + (k_1 + k_3)t + k_2 t^2]}, \quad (4)$$

$$w' = \frac{w(3v - k_3 u - 2k_1 u)}{2u[1 + (k_1 + k_3)t + k_2 t^2]}. \quad (5)$$

Make a transformation

$$h := \sqrt{u\alpha^2 + v\beta^2}, \quad \rho := w\beta, \quad (6)$$

where $u = u(b^2) > 0, v = v(b^2), w = w(b^2) \neq 0$ satisfy the ODEs (3)–(5) such that h is a Riemann metric and the 1-form ρ satisfies certain norm condition with respect to h . There are different choices for u, v, w . It can be shown that, if F defined by (1) is locally projectively flat, then h is of constant sectional curvature μ and ρ is closed and conformal with respect to h (cf. [15] [17]). Then the covariant derivatives $p_{i|j}$ of $\rho = p_i y^i$ with respect to $h = \sqrt{h_{ij} y^i y^j}$ satisfy

$$p_{i|j} = -2ch_{ij} \quad (7)$$

for some scalar function $c = c(x)$. In the whole paper, we will use the data (u, v, w) and (h, μ, c, δ) (related to the choice of (u, v, w)), where δ is defined in Theorem 1.2 (i) below.

Theorem 1.2 *Let (S^n, F) be an $n(\geq 2)$ -dimensional non-Riemannian Finsler manifold, where $F = \alpha\phi(\beta/\alpha)$ is an (α, β) -metric defined by the ODE (1). Assume the conditions (C2) and (C4) in Theorem 1.1 are satisfied. Then the following hold*

- (i) *For every suitable choice of u, v and w , we have $\rho = 2\mu^{-1}c_0$, where $c_0 := c_i y^i = c_{x^i} y^i$. Further, the gradient field ∇c has just two vanishing points $P, Q \in S^n$ and $c = \mu^{-\frac{1}{2}} \delta \cos(\sqrt{\mu} t)$, where t is the arc-length parameter of any geodesic of (S^n, h) connecting P and Q , and $\delta := \sqrt{\|\nabla c\|_h^2 + \mu c^2}$ is a constant.*
- (ii) *For a special suitable choice of u, v and w , the arc-length L of any closed geodesic of (S^n, F) through P, Q and any closed geodesic of (S^n, F) on the hypersurface $c = 0$ has the following expansion as to δ*

$$L = \frac{2\pi}{\sqrt{\mu}} - \frac{4k_3\pi}{\mu^2\sqrt{\mu}} \delta^2 + \frac{4(3k_3^2 + 2k_1k_3 - 2k_2)\pi}{\mu^4\sqrt{\mu}} \delta^4 + o(\delta^6), \quad (8)$$

When $\delta \rightarrow 0$ is small enough, $L > 2\pi/\sqrt{\mu}$ if $k_3 < 0$ or $k_3 = 0$ but $k_2 < 0$; and $L < 2\pi/\sqrt{\mu}$ if $k_3 > 0$ or $k_3 = 0$ but $k_2 > 0$.

- (iii) *The maximal and minimal values of the scalar flag curvature K of F on S^n are just that of the function $R = R(s, t)$ of two variables (s, t) defined on a bounded and closed subset $D = \{(s, t) | 0 \leq t \leq \delta^2/\mu, s^2 \leq B\}$ in the Euclidean plane, where R is defined by*

$$R = \frac{u}{\phi^2 w^2} \left\{ 3u \left(\frac{f_1^2 \phi'^2}{\phi^2} + \frac{4f_1 - f_2^2}{s^2} \right) + 2(u f_3 - v) \left(f_2 - \frac{f_1 s \phi'}{\phi} \right) \right\} t + \frac{\mu u (f_1 s \phi' - f_2 \phi)}{2\phi^3} \quad (9)$$

for every suitable choice of u, v, w (functions of B), in which f_1, f_2, f_3 are defined by

$$f_1 := 1 + (k_1 + k_3)s^2 + k_2 s^4, \quad f_2 := k_2 s^4 - k_1 s^2 - 2, \quad f_3 := 3k_2 s^2 + k_1 + 3k_3,$$

and $B = B(t) := b^2$ is the unique solution of the following equation for $0 \leq t \leq \delta^2/\mu$

$$\frac{w^2(B)B}{u(B) + v(B)B} = \frac{4(\delta^2 - \mu t)}{\mu^2}. \quad (10)$$

Further, the scalar flag curvature K is related to the function R by $K = R(\beta/\alpha, c^2)$.

In the proof of Theorem 1.2 (ii), we will give the integral expression of (8). The expansion (8) is only for two families of closed geodesics of (S^n, F) , and for other closed geodesics of F on S^n , we have not gotten the estimation of their arc-length. For a general metric in Theorem 1.2 (iii), it is difficult to obtain the explicit maximal and minimal values of $R(s, t)$ on D . By a result in [3], it is seen that $\text{Max}_{(s,t) \in D} (R(s, t))$ can never be negative. In Remark 4.2 below, we show a little different function $\tilde{R}(s, t)$ on \tilde{D} , which is equivalent to $R(s, t)$ on D . For some general square metrics in (2), there is a simpler estimation on geodesics and the scalar flag curvature K by a different choice of u, v and w .

Theorem 1.3 *Let (S^n, F) be an $n(\geq 2)$ -dimensional non-Riemannian Finsler manifold, where $F = F_\epsilon^\pm$ is a general square metric defined by (2). Assume the conditions (C2) and (C4) in Theorem 1.1 are satisfied. Let $u = (1 \mp b^2)^2$, $v = 0$, $w = \sqrt{1 \mp b^2}$ in (6). Then the following hold*

(i) α and β are written as

$$\alpha = 4\mu^{-1}(\mu/4 \pm \delta^2\mu^{-1} \mp c^2)h, \quad \beta = 4\mu^{-\frac{3}{2}}\sqrt{\mu/4 \pm \delta^2\mu^{-1} \mp c^2} c_0, \quad (11)$$

(ii) Put $L = L_\epsilon^\pm$ (for $F = F_\epsilon^\pm$) the same meaning as that in Theorem 1.2 (ii). Then

$$L_\epsilon^\pm = \frac{2\pi}{\sqrt{\mu}} \pm \frac{8\pi}{\mu^2\sqrt{\mu}} \delta^2. \quad (12)$$

(iii) Denote by $K = K_\epsilon^\pm$ the scalar flag curvature of F_ϵ^\pm . Then the maximal and minimal values of K_ϵ^\pm in three cases are as follows:

$$\text{Min}(K_2^+) = \frac{(\sqrt{4\delta^2 + \mu^2} - 2\delta)^3}{\mu\sqrt{4\delta^2 + \mu^2}}, \quad \text{Max}(K_2^+) = \frac{(\sqrt{4\delta^2 + \mu^2} + 2\delta)^3}{\mu\sqrt{4\delta^2 + \mu^2}}, \quad (13)$$

$$\text{Min}(K_0^+) = \frac{\mu^2 - 8\delta^2}{\mu}, \quad \text{Max}(K_0^+) = \frac{(\mu^2 + 4\delta^2)^4}{\mu(\mu^2 + 8\delta^2)^3}, \quad (14)$$

$$\text{Min}(K_0^-) = \frac{\mu^5(\mu^2 - 16\delta^2)}{(\mu^2 - 8\delta^2)^3}, \quad \text{Max}(K_0^-) = \frac{(\mu^2 - 4\delta^2)^4}{\mu(\mu^2 - 8\delta^2)^3}. \quad (15)$$

By (13) we have the following corollary for square metrics.

Corollary 1.4 *Any square metric $F = (\alpha + \beta)^2/\alpha$ which is locally projectively flat on S^n always has the scalar flag curvature with positive lower bound.*

By [11] [15] and Corollary 1.4, any square metric of scalar flag curvature on S^n ($n \geq 3$) always has the scalar flag curvature with positive lower bound. Among (α, β) -metrics of non-Randers type which are locally projectively flat on S^n , we have not found any other metric type having the property for square metrics shown in Corollary 1.4. For the metric type F_0^+ , its scalar flag curvature has positive lower bound iff. $\mu^2 > 8\delta^2$ by (14). For the metric type F_0^- (with the regular condition $\mu^2 > 12\delta^2$ by Remark 5.1 below), its scalar flag curvature has positive lower bound iff. $\mu^2 > 16\delta^2$ by (15).

2 Preliminaries

A geodesic $\gamma = \gamma(t) = (x^i(t))$ of a Finsler metric $F = F(x, y)$ are characterized by

$$\frac{d^2 x^i}{dt^2} + 2G^i(x, \frac{dx^i}{dt}) = 0, \quad \left(G^i := \frac{1}{4} g^{il} \{ [F^2]_{x^k y^l} y^k - [F^2]_{x^l} \} \right).$$

In this case, t is the arc-length parameter of γ . A Finsler manifold is called complete (positively complete or negatively complete) if every geodesic $\gamma = \gamma(t)$ is defined on $(-\infty, +\infty)$ ($(0, +\infty)$ or $(-\infty, 0)$) for the arc-length parameter t .

For a Finsler metric F , the Riemann curvature $R_y = R^i_k(y) \frac{\partial}{\partial x^i} \otimes dx^k$ is defined by

$$R^i_k := 2 \frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k}.$$

A Finsler metric F is called of scalar flag curvature if there is a function $K = K(x, y)$ such that

$$R^i_k = KF^2(\delta^i_k - F^{-2}y^i y_k), \quad y_k := (F^2/2)_{y^i y^k} y^i.$$

If K is a constant, then F is called of constant flag curvature.

Two Finsler metrics F and \tilde{F} on a same manifold M are called projectively related if they have same geodesics as point sets, or equivalently their sprays G^i and \tilde{G}^i are related by $\tilde{G}^i = G^i + Py^i$, where $P = P(x, y)$ is called the projective factor satisfying $P(x, \lambda y) = \lambda P(x, y)$ for $\lambda > 0$. A Finsler metric F is said to be locally projectively flat if F is projectively related to a locally Euclidean metric (namely, $G^i = Py^i$ in some local coordinate system everywhere on M). A locally projectively flat Finsler metric ($G^i = Py^i$) is of scalar flag curvature $K = K(x, y)$ which is given by

$$K = \frac{P^2 - P_{x^k} y^k}{F^2}. \quad (16)$$

An (α, β) -metric is a Finsler metric defined by a Riemann metric $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and a 1-form $\beta = b_i(x)y^i$ as follows:

$$F = \alpha\phi(s), \quad s = \beta/\alpha,$$

where $\phi(s) > 0$ is a C^∞ function on $(-b_o, b_o)$. It is proved in [9] that an (α, β) -metric is regular if and only if

$$\phi(s) > 0, \quad \phi(s) - s\phi'(s) > 0, \quad \phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad (|s| \leq b < b_o). \quad (17)$$

For an (α, β) -metric, the spray coefficients G^i of F are given by

$$G^i = G^i_\alpha + \alpha Q s^i_0 + \alpha^{-1} \Theta (-2\alpha Q s_0 + r_{00}) y^i + \Psi (-2\alpha Q s_0 + r_{00}) b^i, \quad (18)$$

where G^i_α denote the spray coefficients of α and

$$Q := \frac{\phi'}{\phi - s\phi'}, \quad \Theta := \frac{Q - sQ'}{2\Delta}, \quad \Psi := \frac{Q'}{2\Delta}, \quad \Delta := 1 + sQ + (b^2 - s^2)Q',$$

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}), \quad s_j := b^i s_{ij}, \quad s^i_j := a^{im} s_{mj},$$

where the covariant derivatives $b_{i|j}$ are taken with respect to the Levi-Civita connection of α , and we define $T_0 := T_i y^i$ for a tensor $T = (T_i)$. For an n -dimensional (α, β) -metric $F = \alpha\phi(\beta/\alpha)$ of non-Randers type, assume β is not parallel with respect to α , and $n \geq 3$ if F is of the metric type F_0^\pm (see (2)). Then F is locally projectively flat iff. ([6] [13])

$$\{1 + (k_1 + k_3)s^2 + k_2 s^4\} \phi''(s) = (k_1 + k_2 s^2) \{\phi(s) - s\phi'(s)\}, \quad (19)$$

$$b_{i|j} = \tau \{ (1 + k_1 b^2) a_{ij} + (k_2 b^2 + k_3) b_i b_j \}, \quad (20)$$

$$G^i_\alpha = \theta y^i - 1/2 \cdot \tau (k_1 \alpha^2 + k_2 \beta^2) b^i, \quad (21)$$

where k_1, k_2, k_3 are constant with $k_2 \neq k_1 k_3$, θ is a 1-form and $\tau = \tau(x)$ is a scalar function. By (20) and (21), define a Riemann metric $h = \sqrt{h_{ij} y^i y^j}$ and a 1-form $\rho_i y^i$ by (6), where

$u = u(b^2) > 0, v = v(b^2), w = w(b^2) \neq 0$ satisfy (3)–(5). Then h is of constant sectional curvature μ and ρ is closed and conformal with respect to h satisfying (7) (cf. [15] [17]). This fact is also verified directly in the proof of Theorem 1.2 (iii) (see (32) below). In [17], it chooses

$$u = e^{2\sigma}, \quad v = (k_1 + k_3 + k_2 b^2)u, \quad w = \sqrt{1 + (k_1 + k_3)b^2 + k_2 b^4} e^\sigma, \quad (22)$$

where σ is defined by

$$\sigma := \frac{1}{2} \int_0^{b^2} \frac{k_2 t + k_3}{1 + (k_1 + k_3)t + k_2 t^2} dt.$$

It is easy to verify that (22) satisfies (3)–(5), and in this case, $\|\beta\|_\alpha = \|\rho\|_h$, namely, the left hand side of (10) is just b^2 .

Proposition 2.1 *Let $F = \alpha\phi(\beta/\alpha)$ be an (α, β) -metric with $\phi(s)$ being defined by (19) with $k_2 \neq k_1 k_3$ and $\phi(0) = 1$. Then F is regular iff. $\phi(s) > 0$ for $|s| \leq b$ and one of the following two cases holds:*

- (i) *If $k_2 \leq 0$, then b^2 is determined by $1 + k_1 b^2 > 0$, $1 + (k_1 + k_3)b^2 + k_2 b^4 > 0$.*
- (ii) *If $k_2 > 0$, then b^2 is determined by $S \cap T$, where S, T are two sets defined by*

$$\begin{aligned} S &:= \{1 + k_1 b^2 > 0, 1 + (k_1 + k_3)b^2 + k_2 b^4 > 0\}, \\ T &:= \{k_1 + k_3 \geq 0\} \cup \{k_1 + k_3 + 2k_2 b^2 \leq 0, 1 + (k_1 + k_3)b^2 + k_2 b^4 > 0\} \\ &\quad \cup \{k_1 + k_3 + 2k_2 b^2 > 0, 4k_2 - (k_1 + k_3)^2 > 0\}. \end{aligned}$$

Proof: By (19) with $\phi(0) = 1$, we have $\phi - s\phi' = \exp\left(-\frac{1}{2} \int_0^{s^2} \frac{k_1 + k_2 \theta}{1 + (k_1 + k_3)\theta + k_2 \theta^2} d\theta\right) > 0$, and

$$\phi - s\phi' + (b^2 - s^2)\phi'' = (\phi - s\phi') \cdot \frac{1 + k_1 b^2 + (k_3 + k_2 b^2)s^2}{1 + (k_1 + k_3)s^2 + k_2 s^4}.$$

The above expression is positive for $|s| \leq b$ iff.

$$1 + k_1 b^2 + (k_3 + k_2 b^2)s^2 > 0, \quad 1 + (k_1 + k_3)s^2 + k_2 s^4 > 0,$$

from which and the regular condition (17), we can easily complete the proof. Q.E.D.

By Proposition 2.1, the regular condition of the metric F_ϵ^\pm defined by (2) can be determined (for some special values of ϵ). By putting $k_1 = \pm 2, k_2 = 0, k_3 = \mp 3$ in (19), we get the metric F_ϵ^\pm . So for F_2^+ and F_0^+ , we obtain the regular condition $b^2 < 1$, and for F_0^- , it is $b^2 < 1/2$.

3 Proof of Theorem 1.1

To prove Theorem 1.1, we need to show some lemmas first as follows.

Lemma 3.1 ([1] [10]) *Let (M, α) be an n -dimensional complete Riemann manifold with $\alpha = \sqrt{a_{ij}y^i y^j}$ being a Riemann metric on the manifold M , and $\tau = \tau(x)$ be a non-constant scalar function on the manifold M . Suppose the covariant derivatives $\tau_{i|j}$ with respect to α satisfy $\tau_{i|j} = \lambda a_{ij}$ for some scalar function $\lambda = \lambda(x)$, where $\tau_i := \tau_{x^i}$. Let $\gamma = \gamma(t)$ be a geodesic tangent to the gradient vector field $\nabla \tau$ with t being the arc-length parameter, and V be a hypersurface (defined by $\tau = \text{constant}$) intersecting with γ at a point P with $\nabla \tau(P) \neq 0$. Then $\tau = \tau(t)$ is a function depending only on t and (M, α) has a line element in the following form*

$$ds^2 = (\tau'(t))^2 d\tilde{s}^2 + dt^2,$$

where $d\tilde{s}^2$ is a line element of the hypersurface V .

Lemma 3.2 Let $\alpha = \sqrt{a_{ij}y^iy^j}$ be a Riemann metric of constant sectional curvature μ and $V = V_iy^i$ be a closed 1-form. Suppose the covariant derivatives $V_{i|j}$ of V with respect to α satisfy $V_{i|j} = -2ca_{ij}$ for some scalar function $c = c(x)$. Then we have

$$c_{i|j} + \mu ca_{ij} = 0, \quad \mu V_i = 2c_i, \quad \|\nabla c\|_\alpha^2 + \mu c^2 = \text{constant}, \quad (23)$$

where ∇ is the gradient operator with respect to α .

Proof : Part of (23) has been proved ([16]). We also show here. By $V_{i|j} = -2ca_{ij}$ and the Ricci identity we have

$$2(c_k a_{ij} - c_i a_{jk}) - V_m R_j^m{}_{ki} = 0,$$

where R is the Riemann curvature tensor of α . Since α is of constant sectional curvature μ , the above equations show

$$2(c_k a_{ij} - c_i a_{jk}) = \mu(V_k a_{ij} - V_i a_{jk}),$$

which are equivalent to

$$(2c_k - \mu V_k) a_{ij} = (2c_i - \mu V_i) a_{jk}.$$

Contracting the above by a^{jk} we obtain $2c_i = \mu V_i$. Thus we get the former two formula in (23).

Now by the first formula in (23) we have

$$(c^i c_i + \mu c^2)_{|k} = 2(c^i c_{i|k} + \mu c c_k) = 2(-\mu c c_k + \mu c c_k) = 0,$$

where $c^i := a^{ij} c_j$. So $\|\nabla c\|_\alpha^2 + \mu c^2$ is a constant.

Q.E.D.

Lemma 3.3 Let (M, F) and (M, \tilde{F}) be two Finsler manifolds with the metric relation $\tilde{F} \leq F$. If (M, \tilde{F}) is complete (resp. positively, or negatively complete), then (M, F) is also complete (resp. positively, or negatively complete).

Proof : Let (M, \tilde{F}) be positively complete. It is easy to see that any forwarded Cauchy sequence induced by the metric d_F is also a forwarded Cauchy sequence induced by the metric $d_{\tilde{F}}$. Then we complete the proof by Hopf-Rinow Theorem for Finsler manifolds. Q.E.D.

Lemma 3.4 Let $F = \alpha\phi(\beta/\alpha)$ satisfy the condition (C5) in Theorem 1.1, where $\phi(s)$ is determined by (19) with $k_2 \neq k_1 k_3$. Define h and ρ by (6), where u, v, w is given by (22). Then F is positively complete iff. α is complete iff. h is complete.

Proof : By the condition (C5) $b(x_0) = \sup_{x \in M} b(x)$ at some point $x_0 \in M$, and the continuity of the positive function $\phi(s)$ on $|s| \leq b(x_0)$, we get two constants $m_1 > 0$ and $M_1 > 0$ such that $m_1 \alpha \leq F \leq M_1 \alpha$. So F is positively complete iff. α is complete by Lemma 3.3.

By the choice of u, v, w given by (22), we have

$$1 + \frac{v}{u} \frac{\beta^2}{\alpha^2} = 1 + \frac{v}{u} s^2 = 1 + (k_1 + k_3 + k_2 b^2) s^2.$$

Therefore we get

$$k_1 + k_3 + k_2 b^2 \geq 0 : \quad 1 \leq 1 + \frac{v}{u} \frac{\beta^2}{\alpha^2} \leq 1 + (k_1 + k_3 + k_2 b^2) b^2,$$

$$k_1 + k_3 + k_2 b^2 < 0 : \quad 1 + (k_1 + k_3 + k_2 b^2) b^2 \leq 1 + \frac{v}{u} \frac{\beta^2}{\alpha^2} \leq 1.$$

Now put

$$A_1(b^2) := \min\{1, 1 + (k_1 + k_3 + k_2 b^2) b^2\}, \quad A_2(b^2) := \max\{1, 1 + (k_1 + k_3 + k_2 b^2) b^2\}.$$

Then it follows from the regular condition of F shown in Proposition 2.1 that both $A_1(t)$ and $A_2(t)$ are positive and continuous functions on the closed interval $|t| \leq b(x_0)$. Let

$$m_2 := \min(\sqrt{u(t)A_1(t)}, |t| \leq b(x_0)), \quad M_2 := \max(\sqrt{u(t)A_2(t)}, |t| \leq b(x_0))$$

Then by continuity, we have $M_2 \geq m_2 > 0$. Therefore, we obtain

$$m_2 \alpha \leq h = \sqrt{u(1 + \frac{v}{u} \frac{\beta^2}{\alpha^2})} \cdot \alpha \leq M_2 \alpha,$$

which implies that α is complete iff. h is complete by Lemma 3.3. Q.E.D.

Lemma 3.5 *Let $F = \alpha\phi(\beta/\alpha)$ satisfy (19)–(21). Then F is projectively related to the Riemann metric h defined by (6), where u, v are determined by (3)–(5).*

Proof : In some local coordinate, the spray G_α^i are given by (21). Plugging (20), (21) and

$$s_0 = 0, \quad s^i_0 = 0, \quad \Psi = \frac{1}{2} \frac{\phi''}{\phi - s\phi' + (b^2 - s^2)\phi''}$$

into (18), and then using (19), we easily get the spray G^i of F in the form $G^i = P_1 y^i$ for some positively homogeneous function P_1 of degree one on TM . Now define a Riemann metric h shown in (6). In the same local coordinate as shown in (21) for G_α^i , using (20), (21) and (3)–(5), one direct computation shows that the spray G_h^i of h satisfy $G_h^i = P_2 y^i$ for some positively homogeneous function P_2 of degree one on TM . Therefore, the sprays G^i and G_h^i satisfy the relation $G^i = G_h^i + P y^i$ for $P := P_1 - P_2$ under any local coordinate. Thus F and h are projectively related. Q.E.D.

Now we start the proof of Theorem 1.1. Let (M, F) be an $n(\geq 2)$ -dimensional Finsler manifold, where $F = \alpha\phi(\beta/\alpha)$ is an (α, β) -metric on M . If β is parallel with respect to α , then by (18) we get $G^i = G_\alpha^i$. Thus α is of constant sectional curvature since F is locally projectively flat. If $\beta = 0$, then $F = \alpha$ is Riemannian. If $\beta \neq 0$, then it is easy to show that α is flat, and so F is flat-parallel.

Suppose β is not parallel with respect to α . Then by the conditions (C1), (C2) and (C4), the (α, β) -metric $F = \alpha\phi(\beta/\alpha)$ satisfies (19)–(21) with $k_2 \neq k_1 k_3$ (see [6] [13]). Now make a transformation (6), namely,

$$h := \sqrt{u\alpha^2 + v\beta^2}, \quad \rho := w\beta,$$

where $u = u(b^2) > 0, v = v(b^2), w = w(b^2) \neq 0$ are suitable functions satisfying the ODEs (3)–(5). As shown in Section 2, $h = \sqrt{h_{ij}y^i y^j}$ is a Riemann metric of constant sectional curvature μ and the 1-form $\rho = p_i y^i$ is closed and conformal with respect to h (cf. [15] [17]). So the covariant derivatives $p_{i|j}$ of ρ with respect to h satisfy (7), that is,

$$p_{i|j} = -2ch_{ij} \tag{24}$$

for some scalar function $c = c(x)$. Then by Lemma 3.2, the scalar function c in (24) satisfies

$$c_{i|j} + \mu ch_{ij} = 0. \tag{25}$$

Now we fix a special choice of u, v, w given by (22), namely,

$$u = e^{2\sigma}, \quad v = (k_1 + k_3 + k_2 b^2)u, \quad w = \sqrt{1 + (k_1 + k_3)b^2 + k_2 b^4} e^\sigma, \\ \sigma := \frac{1}{2} \int_0^{b^2} \frac{k_2 t + k_3}{1 + (k_1 + k_3)t + k_2 t^2} dt.$$

Note that in this case, $b^2 = \|\beta\|_\alpha^2 = \|\rho\|_h^2$, and h is complete by Lemma 3.4.

Case I: Assume $c = \text{constant}$ on the manifold M . We will prove $c = 0$ in this case.

Suppose $c \neq 0$. Then by (25) we get $\mu = 0$, that is, h is locally Euclidean. Since (M, h) is complete by Lemma 3.4, its universal covering metric space is (R^n, \tilde{h}) which is locally isometric with (M, h) , where $\tilde{h} = |y|$. The lift $\tilde{\rho}$ of ρ in (R^n, \tilde{h}) satisfies $\tilde{p}_{i|j} = -2c\tilde{h}_{ij}$ (see (24)), from which it is easy to get $\tilde{p}_i = \tilde{p}^i = -2cx^i + \xi^i$ under the global coordinate (x^i) of \tilde{h} , where ξ is a constant vector. But $b^2 = \|\beta\|_\alpha^2 = \|\rho\|_h^2 \leq b^2(x_0)$ is bounded, so

$$\|\rho\|_h^2 = \|\tilde{\rho}\|_{\tilde{h}}^2 = 4c^2|x|^2 - 4c\langle \xi, x \rangle + |\xi|^2$$

is also bounded on R^n . However, it is unbounded since $c \neq 0$, which is a contradiction. Thus we get $c = 0$ if $c = \text{constant}$ on the manifold M .

Now by $c = 0$, it follows from (24) that ρ is parallel with respect to h . If $\rho = 0$, it is clear that $F = \alpha$. If $\rho \neq 0$, then h is locally flat. Now by (6) we get

$$\alpha^2 = \frac{1}{u}h^2 - \frac{v}{uw^2}\rho^2, \quad \beta = \frac{1}{w}\rho.$$

The above shows that α is flat and β is parallel with respect to α , since locally α and β are independent of $x \in M$, which follows from the fact that $u = u(b^2), v = v(b^2), w = w(b^2)$ are constant and locally $h = |y|$ and $\rho = \langle \xi, y \rangle$ for some constant vector ξ .

Case II: Assume c is a non-constant. In this case, we will show that $\mu > 0$ and the manifold M is homeomorphic to the n -sphere S^n .

If $\mu = 0$, then by (23) in Lemma 3.2, we have $2c_i = \mu p_i$. So $c_i = 0$, which shows that c is a constant. This is a contradiction.

Now assume $\mu \neq 0$. It is clear that (25) implies that the trajectories of the gradient vector field ∇c are geodesics of h as point sets. Let $x(t)$ be a geodesic of (M, h) tangent to ∇c , where t is the arc-length parameter. Put $c(t) = c(x(t))$. Then by (25) we have

$$\frac{d^2 c(t)}{dt^2} + \mu c(t) = 0. \quad (26)$$

If $\mu < 0$, then the solutions of (26) are given by

$$c(t) = ke^{\sqrt{-\mu}t} + le^{-\sqrt{-\mu}t}, \quad (27)$$

where k, l are constant with $k^2 + l^2 \neq 0$. Since (M, h) is complete by Lemma 3.4, $c(t)$ in (27) is defined on $(-\infty, +\infty)$. By (23) in Lemma 3.2, we have $2c_i = \mu p_i$. By Lemma 3.1, we get $\|\nabla c\|_h^2 = (c'(t))^2$. So $b^2 = \|\rho\|_h^2 = 4\mu^{-2}(c'(t))^2$ is unbounded for $t \in (-\infty, +\infty)$ by (27). But by the condition (C5) in Theorem 1.1, b^2 is bounded on M . We get a contradiction.

The above discussion shows that there must have $\mu > 0$. Since c is a non-constant function satisfying (25) and (M, h) is complete, it follows from a result in [4] [10] that the manifold M is homeomorphic to the n -sphere S^n .

The discussions in Case I and Case II above have actually proved the following lemma.

Lemma 3.6 *Let $h = \sqrt{h_{ij}y^i y^j}$ be a complete Riemannian metric of constant sectional curvature μ and the 1-form $\rho = p_i y^i$ be closed and conformal with respect to h satisfying (24) for some scalar function $c = c(x)$. Suppose $\|\rho\|_h$ is bounded. Then $\mu = 0$ and ρ is parallel with respect to h , or $\mu > 0$ and M is homeomorphic to the n -sphere S^n .*

Now we prove the final part in Theorem 1.1, namely, if F satisfies the conditions (C3)–(C5), and (C6) F is of constant flag curvature, then $F = \alpha$ or F is flat-parallel. According to the results in [2] [14], if an (α, β) -metric $F = \alpha\phi(\beta/\alpha)$ is locally projectively flat with constant flag curvature, then F is one of the three classes: (a1) F is flat-parallel; (a2) $F = \alpha + \beta$ is of Randers type; (a3) $F = (\alpha + \beta)^2/\alpha$ is of square type.

For a Randers metric $F = \alpha + \beta$ (the regular condition $b^2 < 1$) which is locally projectively flat with constant flag curvature, it follows from [8] that α is of non-positive constant sectional curvature $\mu \leq 0$ and the covariant derivative of β with respect to α satisfies

$$b_{i;j} = 2k(a_{ij} - b_i b_j), \quad (28)$$

for some constant k . Define a Riemann metric $h = \sqrt{h_{ij}y^i y^j}$ and 1-form $\rho = p_i y^i$ by

$$h := \alpha, \quad \rho := \frac{1}{\sqrt{1-b^2}}\beta.$$

Under the above transformation, it can be directly verified that $\|\rho\|_h^2 = b^2/(1-b^2)$, and (28) is equivalent to

$$p_{i;j} = -2ch_{ij}, \quad (c := -\frac{k}{\sqrt{1-b^2}}),$$

where the covariant derivative is taken with respect to $h = \alpha$. Thus (24) holds for some scalar function $c = c(x)$. Further, since $F = \alpha + \beta \leq 2\alpha$, it follows from Lemma 3.3 that $h = \alpha$ is complete. Now it follows from the condition (C5) and $\|\rho\|_h^2 = b^2/(1-b^2)$ that $\|\rho\|_h$ is bounded. Plus the fact $\mu \leq 0$, it directly follows from Lemma 3.6 that $F = \alpha$ or F is flat-parallel.

For a square metric $F = (\alpha + \beta)^2/\alpha$ (the regular condition $b^2 < 1$) which is locally projectively flat with constant flag curvature, consider a special transformation of (6) to define a new pair (h, ρ) by putting $u = (1-b)^2, v = 0$, and $w = \sqrt{1-b^2}$, namely,

$$h := (1-b^2)\alpha, \quad \rho := \sqrt{1-b^2}\beta.$$

it is already shown that h is of constant sectional curvature μ and ρ is a closed and conformal 1-form with respect to h satisfying (24) for some scalar function $c = c(x)$. In [12], it actually proves $\mu \leq 0$ (also see [15]). Then it follows from the condition (C5) and $\|\rho\|_h^2 = b^2/(1-b^2)$ that $\|\rho\|_h$ is bounded. Plus the fact $\mu \leq 0$, it directly follows from Lemma 3.6 that $F = \alpha$ or F is flat-parallel. Q.E.D.

Remark 3.7 Let $\phi(s)$ be a fixed function with $|s| < b_o$ (possibly $b_o = +\infty$). Let $\hat{b} \leq b_o$ be a constant such that for arbitrary α and β with $k < \|\beta\|_\alpha < \hat{b}$ (where k is a constant with $0 \leq k < \hat{b}$), $F = \alpha\phi(\beta/\alpha)$ is always regular, and on the other hand, if a pair of α and β satisfies $\|\beta\|_\alpha \geq \hat{b}$, then $F = \alpha\phi(\beta/\alpha)$ is non-regular. Note that \hat{b} is dependent on the function ϕ and independent of α, β . For example, if $\phi(s) = 1 + s$ or $(1+s)^2$ or $1 + s^2$, we have $\hat{b} = 1$; if $\phi(s) = 1 - s^2$, we have $\hat{b} = 1/\sqrt{2}$.

Now in Theorem 1.1, the condition (C5) can be replaced by a weaker condition $b = \|\beta\|_\alpha \leq \bar{b} < \hat{b}$ for a constant \bar{b} .

4 Proof of Theorem 1.2

Let $\phi(s)$ satisfy the ODE (1), and $F = \alpha\phi(\beta/\alpha)$ is an (α, β) -metric on S^n . Assume the conditions (C2) and (C4) in Theorem 1.1 hold. Then β is not parallel with respect to α . Otherwise, α is flat on S^n since $\beta \neq 0$ (F is non-Riemannian), which is impossible. Now as shown in the proof of Theorem 1.1, define a pair (h, ρ) (see (6)) for every suitable u, v, w

satisfying (3)–(5). Then $h = \sqrt{h_{ij}y^i y^j}$ is a Riemann metric of constant sectional curvature $\mu > 0$ and $\rho = p_i y^i$ is a closed and conformal 1-form with respect to h satisfying (24) and then (25) for a scalar function $c = c(x)$.

By Lemma 3.2, we have $\rho = 2\mu^{-1}c_0$ and $\delta := \sqrt{\|\nabla c\|_h^2 + \mu c^2}$ is a constant. Put $c(t) = c(x(t))$, where $x(t)$ is a geodesic of (M, h) tangent to ∇c with t being the arc-length parameter. Solving the ODE (26) for $\mu > 0$ we obtain $c = \mu^{-\frac{1}{2}}\delta \cos(\sqrt{\mu} t)$ for a suitably chosen again arc-length parameter t . By Lemma 3.1, we have $\|\nabla c\|_h = |c'(t)|$. Thus the gradient field ∇c has just two vanishing points $P, Q \in S^n$. This completes the proof of item (i) of Theorem 1.2.

To prove item (ii) of Theorem 1.2, we choose u, v, w given by (22). By Lemma 3.5, the (α, β) -metric F is projectively related to the Riemann metric h . So F and h have same geodesics as point sets. In the following we consider the F -length of two families of geodesics of F .

The two vanishing points P, Q of the gradient field ∇c are just a pair of antipodal points on S^n . Let $x(t)$ be an arbitrary closed geodesic of h connecting P and Q (great circle of h) with $0 \leq t \leq 2\pi/\sqrt{\mu}$, and $c = \mu^{-\frac{1}{2}}\delta \cos(\sqrt{\mu} t)$ along $x(t)$. From (6) and $\rho = 2\mu^{-1}c_0$, we have

$$\alpha^2 = u^{-1}[h_\mu^2 - 4(\mu w)^{-2}v c_0^2], \quad \beta = 2(\mu w)^{-1}c_0. \quad (29)$$

Then along the closed geodesic $x(t)$ of h , from $h(x(t), x'(t)) = 1$, $c_0(x(t), x'(t)) = c'(t)$, $b^2 = \|\rho\|_h^2 = 4\mu^{-2}(c'(t))^2$ and the choice of u, v, w defined by (22), we obtain from (29)

$$\begin{aligned} \alpha(x(t), x'(t)) &= \frac{1}{w} \sqrt{\frac{w^2 - b^2 v}{u}} = \frac{1}{w} = \frac{e^{-\frac{1}{2} \int_0^t \frac{k_2 \theta + k_3}{1 + (k_1 + k_3)\theta + k_2 \theta^2} d\theta}}{\sqrt{1 + (k_1 + k_3)b^2 + k_2 b^4}}, \\ \frac{\beta(x(t), x'(t))}{\alpha(x(t), x'(t))} &= \frac{2\mu^{-1}w^{-1}\sqrt{u} c'(t)}{\sqrt{1 - 4\mu^{-2}w^{-2}v(c'(t))^2}} = \frac{2\mu^{-1}c'(t)}{\sqrt{u^{-1}(w^2 - b^2 v)}} = 2\mu^{-1}c'(t). \end{aligned}$$

Therefore, the F -length L_1 of the geodesic $x(t)$ is given by

$$L_1 = \int_0^{\frac{2\pi}{\sqrt{\mu}}} \alpha(x(t), x'(t)) \phi\left(\frac{\beta(x(t), x'(t))}{\alpha(x(t), x'(t))}\right) dt = \int_0^{\frac{2\pi}{\sqrt{\mu}}} \frac{\phi(-2\mu^{-1}\delta \sin(\sqrt{\mu} t))}{\sqrt{1 + (k_1 + k_3)B + k_2 B^2}} e^\sigma dt, \quad (30)$$

where

$$B := 4\mu^{-2}\delta^2 \sin^2(\sqrt{\mu} t), \quad \sigma := \frac{1}{2} \int_0^B \frac{k_2 \theta + k_3}{1 + (k_1 + k_3)\theta + k_2 \theta^2} d\theta,$$

By the ODE (19) on $\phi(s)$, we easily get

$$\phi(s) = 1 + \phi'(0)s + \frac{1}{2}k_1 s^2 + \left(\frac{1}{12}k_2 - \frac{1}{8}k_1^2 - \frac{1}{12}k_1 k_3\right)s^4 + o(s^6).$$

Using the above expansion on $\phi(s)$, expanding (30) as to δ at $\delta = 0$ we obtain (8).

Now we compute the F -length of the family of closed geodesics lying on the hypersurface $c = 0$. It is clear that the hypersurface $c = 0$ is totally geodesic in (S^n, h) . Let $z(t)$ is a closed geodesic of (S^n, h) lying on the hypersurface $c = 0$, where t is the arc-length parameter with respect to h . Then $z(t)$ is also a geodesic of (S^n, F) as a point set. It is easily seen that $c_0(z(t), z'(t)) = 0$, and $b^2(z(t)) = 4\mu^{-2}\delta^2$. Thus it follows from (29) and then from (22) that the F -length L_2 of $z(t)$ is given by

$$\begin{aligned} L_2 &= \int_0^{\frac{2\pi}{\sqrt{\mu}}} \frac{1}{\sqrt{u}} dt = \int_0^{\frac{2\pi}{\sqrt{\mu}}} \exp\left(-\frac{1}{2} \int_0^{\frac{4\delta^2}{\mu^2}} \frac{k_2 \theta + k_3}{1 + (k_1 + k_3)\theta + k_2 \theta^2} d\theta\right) dt, \\ &= \frac{2\pi}{\sqrt{\mu}} \exp\left(-\frac{1}{2} \int_0^{\frac{4\delta^2}{\mu^2}} \frac{k_2 \theta + k_3}{1 + (k_1 + k_3)\theta + k_2 \theta^2} d\theta\right). \end{aligned} \quad (31)$$

Now expanding (31) as to δ at $\delta = 0$ we also obtain (8).

Remark 4.1 *The two integrals given by (30) and (31) should be equal. But we have not found a direct way to prove it.*

To show the estimation in (iii) for the maximal and minimal values of the scalar flag curvature K , we first compute the expression of K . Since F defined by (19) with $k_2 \neq k_1 k_3$ satisfies the conditions (C2) and (C4) in Theorem 1.1, we have (20) and (21). Define a Riemann metric $h = \sqrt{h_{ij}y^i y^j}$ and 1-form $\rho = p_i y^i$ by (6), where $u = u(b^2) > 0, v = v(b^2), w = w(b^2) \neq 0$ are arbitrary suitable functions satisfying the ODEs (3)–(5). Now using (3)–(5), it follows from (20) and (21) that

$$G_h^i = \left(\frac{(v - k_1 u)\tau}{u} \beta + \theta \right) y^i, \quad p_{i|j} = \frac{w\tau}{u} h_{ij} = -2ch_{ij}, \quad (32)$$

where the covariant derivatives are taken with respect to h . So by (32), h is locally projectively flat (equivalently, h is of constant sectional curvature μ) and ρ is closed and conformal with respect to h (cf. [15] [17]). In some local coordinate, h and its spray are given by

$$h = \frac{\sqrt{(1 + \mu|x|^2)|y|^2 - \mu\langle x, y \rangle^2}}{1 + \mu|x|^2}, \quad G_h^i = -\frac{\mu\langle x, y \rangle}{1 + \mu|x|^2} y^i. \quad (33)$$

Then the function c in (32) is locally given by

$$c = \frac{-k + \mu\langle \xi, x \rangle}{2\sqrt{1 + \mu|x|^2}}, \quad (34)$$

where k is a constant and ξ is a vector. By (32)–(34) we obtain

$$\theta = \frac{(k_1 u - v)\tau}{u} \beta - \frac{\mu\langle x, y \rangle}{1 + \mu|x|^2}, \quad \tau = \frac{k - \mu\langle \xi, x \rangle}{\sqrt{1 + \mu|x|^2}} \frac{u}{w}. \quad (35)$$

Put $\theta_0 = \theta_i y^i = \theta_{x^i} y^i$ and $\tau_0 = \tau_i y^i = \tau_{x^i} y^i$. Then by (35) we get

$$\begin{aligned} \theta_0 &= \frac{k_1 u - v}{u} (\tau^2 \alpha^2 + \tau_0 \beta + 2\tau \theta \beta) + \left\{ k_1 k_3 - 2k_2 + (k_3 + 2k_1) \frac{v}{u} - \frac{2v^2}{u^2} \right\} \tau^2 \beta^2 \\ &\quad - \frac{\mu[(1 + \mu|x|^2)|y|^2 - 2\mu\langle x, y \rangle^2]}{(1 + \mu|x|^2)^2}, \end{aligned} \quad (36)$$

$$\tau_0 = \frac{\tau(k_3 u - v)(k - \mu\langle \xi, x \rangle)}{w\sqrt{1 + \mu|x|^2}} \beta - \frac{\mu u}{\sqrt{1 + \mu|x|^2}} \left\{ \frac{(k - \mu\langle \xi, x \rangle)\langle x, y \rangle}{w(1 + \mu|x|^2)} + \langle \xi, y \rangle \right\}, \quad (37)$$

where we have used (20), (21), the ODEs (3)–(5) and

$$u_{x^i} y^i = 2u'(b^2)r_0, \quad v_{x^i} y^i = 2v'(b^2)r_0, \quad \beta_{x^i} y^i = r_{00} + 2b_m G_\alpha^m.$$

Plugging (19)–(21) into (18) we obtain $G^i = P y^i$, where P is given by ([2])

$$P = \theta + \frac{1}{2} \tau \left\{ \frac{1 + (k_1 + k_3)s^2 + k_2 s^4}{\phi} \phi' - (k_1 + k_2 s^2)s \right\} \alpha,$$

and then by (16), we obtain the scalar flag curvature K given by

$$\begin{aligned} K &= \frac{\theta^2 - \theta_0}{\alpha^2 \phi^2} + \frac{1}{2\phi^2} \left\{ (k_1 + k_2 s^2)s - [1 + (k_1 + k_3)s^2 + k_2 s^4] \frac{\phi'}{\phi} \right\} \frac{\tau_0}{\alpha} \\ &\quad - \frac{[1 + (k_1 + k_3)s^2 + k_2 s^4](k_1 + 2k_3 + 3k_2 s^2)s\tau^2 \phi'}{2\phi^3} + [1 + (k_1 + k_3)s^2 + k_2 s^4]^2 \times \\ &\quad \frac{3\tau^2 (\phi')^2}{4\phi^4} + \frac{[4k_2 - k_1^2 + 2k_2(k_1 + 2k_3)s^2 + 3k_2^2 s^4]s^2 \tau^2}{4\phi^2}. \end{aligned} \quad (38)$$

By Lemma 3.2 we have $2c_0 = \mu\rho$ (see (23)), and so $2c_0 = \mu w\beta$. Using this fact, differentiating both sides of (34) with respect to x^i gives

$$(k - \mu\langle\xi, x\rangle)\langle x, y\rangle + (1 + \mu|x|^2)[\langle\xi, y\rangle - w\sqrt{1 + \mu|x|^2}\beta] = 0. \quad (39)$$

By $h = \sqrt{u\alpha^2 + v\beta^2}$ and (33) we get

$$\frac{(1 + \mu|x|^2)|y|^2 - \mu\langle x, y\rangle^2}{(1 + \mu|x|^2)^2} = u\alpha^2 + v\beta^2. \quad (40)$$

Now plugging the expressions of θ, τ and θ_0, τ_0 given by (35)–(37) into the scalar flag curvature K shown in (38), we can obtain the local expression of K under a local coordinate system shown in (33) for h . Then using (34), (39) and (40), the scalar flag curvature K can be further written in the form $R(\beta/\alpha, c^2)$, where $R(s, t)$ is a function of two variables (s, t) defined by (9).

Next we show that $R(s, t)$ is defined on a bounded and closed subset in the Euclidean plane. Since $\rho = 2\mu^{-1}c_0$ by Lemma 3.2, we have

$$\|\rho\|_h^2 = 4\mu^{-2}\|\nabla c\|_h^2 = 4\mu^{-2}(\delta^2 - \mu c^2).$$

Since h and ρ are defined by α and β in (6), we easily get

$$\|\rho\|_h^2 = \frac{w^2(b^2)b^2}{u(b^2) + v(b^2)b^2}.$$

Thus we obtain (10), namely,

$$\frac{w^2(B)B}{u(B) + v(B)B} = \frac{4(\delta^2 - \mu t)}{\mu^2}, \quad (41)$$

where $B := b^2$ and $t := c^2$. Note that the function on the left hand side of (41) is strictly increasing on the variable B , which follows from

$$\frac{d}{dB} \left(\frac{w^2(B)B}{u(B) + v(B)B} \right) = \frac{w^2}{u[1 + (k_1 + k_3)B + k_2B^2]} > 0$$

because of the ODEs (3)–(5) and $1 + (k_1 + k_3)b^2 + k_2b^4 > 0$ by the regular condition shown in Proposition 2.1. So in the equation (41), B is uniquely determined for every t , and thus $B = B(t)$ is a function of t . Now since $0 \leq c^2 \leq \delta^2/\mu$ by Theorem 1.2 (i) and $|\beta/\alpha| \leq b$, we see $R(s, t)$ is defined on the bounded and closed subset $D = \{(s, t) | 0 \leq t \leq \delta^2/\mu, s^2 \leq B\}$ in the Euclidean plane, where $B = B(t)$ is determined by the equation (41). Q.E.D.

Remark 4.2 In Theorem 1.2 (iii), we can rewrite the function $R = R(s, t)$ of two variables as a different function $\tilde{R} = \tilde{R}(s, t)$ of two variables. Put

$$\begin{aligned} \tilde{R} &= \frac{u}{\phi^2 w^2} \left\{ 3u \left(\frac{f_1^2 \phi'^2}{\phi^2} + \frac{4f_1 - f_2^2}{s^2} \right) + 2(u f_3 - v) \left(f_2 - \frac{f_1 s \phi'}{\phi} \right) \right\} A + \frac{\mu u (f_1 s \phi' - f_2 \phi)}{2\phi^3}, \quad (42) \\ &\quad \left(A := \frac{1}{\mu} \left(\delta^2 - \frac{\mu^2}{4} \frac{w^2 t^2}{u + v t^2} \right) \text{ by (41)} \right), \end{aligned}$$

where u, v, w are functions of t^2 and $(s, t) \in \tilde{D}$ with $\tilde{D} = \{(s, t) | 0 \leq t \leq t_o, |s| \leq t\}$, in which t_o is the unique positive constant satisfying

$$\frac{w^2(t_o^2)t_o^2}{u(t_o^2) + v(t_o^2)t_o^2} = \frac{4\delta^2}{\mu^2},$$

5 Proof of Theorem 1.3

Let (S^n, F) be an $n(\geq 2)$ -dimensional non-Riemannian Finsler manifold with $F = F_\epsilon^\pm$ defined by (2), and the conditions (C2) and (C4) in Theorem 1.1 be satisfied. As shown in the proof of Theorem 1.2, β is not parallel with respect to α . Since $\phi(s) = 1 + \epsilon s \pm s^2$ for the metric $F = F_\epsilon^\pm$ defined in (2), we may put $k_1 = \pm 2, k_2 = 0, k_3 = \mp 3$ in (3)–(5). So in this case, we may choose

$$u = (1 \mp B)^2, \quad v = 0, \quad w = \sqrt{1 \mp B}, \quad (B := b^2). \quad (43)$$

Then using (43), we get a pair (h, ρ) by (6), where h is a Riemann metric of constant sectional curvature $\mu > 0$ and ρ is a closed and conformal 1-form. Correspondingly we get a scalar function $c = c(x)$ satisfying (7) and a positive constant δ defined by c . By the choice of u, v, w shown in (43), it follows from (41) (therein $t = c^2$) that

$$B = \frac{4(\delta^2 - \mu c^2)}{\mu^2 \pm 4(\delta^2 - \mu c^2)}. \quad (44)$$

Now for the proof of item (i) in this theorem, using (43) and (44), it is clear from (6) that α and β can be expressed by h, μ, c, δ shown in (11), namely,

$$\alpha = 4\mu^{-1}(\mu/4 \pm \delta^2\mu^{-1} \mp c^2)h, \quad \beta = 4\mu^{-\frac{3}{2}}\sqrt{\mu/4 \pm \delta^2\mu^{-1} \mp c^2} c_0. \quad (45)$$

The proof of Theorem 1.3 (ii) is similar to that of Theorem 1.2 (ii). Let $x(t)$ ($0 \leq t \leq 2\pi/\sqrt{\mu}$) be an arbitrary closed geodesic of h connecting the two points P, Q on which the gradient ∇c vanishes. Along $x(t)$, we have

$$c = \mu^{-\frac{1}{2}}\delta \cos(\sqrt{\mu} t), \quad h(x(t), x'(t)) = 1, \quad c_0(x(t), x'(t)) = c'(t).$$

Since β is closed, it follows from (45) that the F -length L_1 of the geodesic $x(t)$ is given by

$$\begin{aligned} L_1 &= \int_0^{\frac{2\pi}{\sqrt{\mu}}} F_\epsilon^\pm(x(t), x'(t)) dt = \int_0^{\frac{2\pi}{\sqrt{\mu}}} F_0^\pm(x(t), x'(t)) dt \\ &= \int_0^{\frac{2\pi}{\sqrt{\mu}}} \left\{ \alpha(x(t), x'(t)) \pm \frac{\beta^2(x(t), x'(t))}{\alpha(x(t), x'(t))} \right\} dt \\ &= \int_0^{\frac{2\pi}{\sqrt{\mu}}} \left\{ 1 \pm \frac{4\delta^2}{\mu^2} \mp \frac{4\delta^2}{\mu^2} \cos(2\sqrt{\mu} t) \right\} dt \\ &= \frac{2\pi}{\sqrt{\mu}} \pm \frac{8\pi}{\mu^2\sqrt{\mu}} \delta^2. \end{aligned}$$

Thus we obtain (12) for the F -length of the family of closed geodesics collecting P, Q .

Now let $z(t)$ ($0 \leq t \leq 2\pi/\sqrt{\mu}$) be an arbitrary closed geodesic of (S^n, h) lying on the hypersurface $c = 0$. By $c(z(t)) = 0$, $c_0(z(t), z'(t)) = 0$ and $h(z(t), z'(t)) = 1$, it is easily seen that the F -length L_2 of the geodesic $z(t)$ is given by

$$\begin{aligned} L_2 &= \int_0^{\frac{2\pi}{\sqrt{\mu}}} F_\epsilon^\pm(x(t), x'(t)) dt = \int_0^{\frac{2\pi}{\sqrt{\mu}}} \alpha(z(t), z'(t)) dt \\ &= \int_0^{\frac{2\pi}{\sqrt{\mu}}} \left(1 \pm \frac{4\delta^2}{\mu^2} \right) dt = \frac{2\pi}{\sqrt{\mu}} \pm \frac{8\pi}{\mu^2\sqrt{\mu}} \delta^2. \end{aligned}$$

This gives (12) for the F -length of the closed geodesics lying on the hypersurface $c = 0$.

Finally, we come to the proof of item (iii) of this theorem. Plug $k_1 = \pm 2, k_2 = 0, k_3 = \mp 3$, (43) and (44) into the scalar flag curvature $K_\epsilon^\pm = R(\beta/\alpha, c^2)$ for $F = F_\epsilon^\pm$, where $R = R(s, t)$ is given by (9), and then we obtain

$$K_\epsilon^\pm = \frac{6\mu(\epsilon^2 \mp 4)(1 \mp s^2)^2 c^2 + (\mu^2 \pm 4\delta^2)(\pm \epsilon s^3 \pm 6s^2 + 3\epsilon s + 2)(1 + \epsilon s \pm s^2)}{128\mu^{-2}(\mu/4 \pm \delta^2\mu^{-1} \mp c^2)^3(1 + \epsilon s \pm s^2)^4}. \quad (46)$$

Case (A) : For $F = F_2^+$, by (46) we get

$$K_2^+ = \frac{\xi\mu^3}{16}[(1+s)(\xi - c^2)]^{-3}, \quad (\xi := \delta^2\mu^{-1} + \mu/4).$$

By Theorem 1.2 (iii), there is a function $R = R(s, t)$ of two variables such that $K_2^+ = R(\beta/\alpha, c^2)$. By (42) in Remark 4.2, R can be written as $\tilde{R} = \tilde{R}(s, t)$ defined on \tilde{D} with

$$\begin{aligned} \tilde{R} &= \frac{\xi\mu^3}{16}[(1+s) \frac{\mu}{4(1-t^2)}]^{-3} = 4\xi \left(\frac{1+s}{1-t^2}\right)^{-3}, \\ \tilde{D} &= \{(s, t) | 0 \leq t \leq t_o, \quad |s| \leq t\}, \quad (t_o := \frac{2\delta}{\sqrt{4\delta^2 + \mu^2}}). \end{aligned}$$

It is clear that

$$\begin{aligned} \text{Min}_{(s,t) \in \tilde{D}} \left(\frac{1+s}{1-t^2}\right) &= \text{Min}_{t \in [0, t_o]} \left(\frac{1}{1+t}\right) = \frac{1}{1+t_o}, \\ \text{Max}_{(s,t) \in \tilde{D}} \left(\frac{1+s}{1-t^2}\right) &= \text{Max}_{t \in [0, t_o]} \left(\frac{1}{1-t}\right) = \frac{1}{1-t_o}. \end{aligned}$$

Now we can easily obtain the maximal and minimal values of K_2^+ on S^n given by (13).

Case (B) : For $F = F_0^+$, by (46) we get

$$K_0^+ = \frac{\mu^5[-12\mu(1-s^2)^2 c^2 + (\mu^2 + 4\delta^2)(1+s^2)(1+3s^2)]}{(\mu^2 + 4\delta^2 - 4\mu c^2)^3(1+s^2)^4}.$$

Then in Remark 4.2, $\tilde{R} = \tilde{R}(s, t)$ and \tilde{D} are given by

$$\tilde{R} = \frac{(1-t^2)^2[2(\mu^2 + 4\delta^2)(1-5s^2)t^2 + 3\mu^2 s^4 + 4(\mu^2 + 10\delta^2)s^2 + \mu^2 - 8\delta^2]}{\mu(1+s^2)^4}, \quad (47)$$

$$\tilde{D} = \{(s, t) | 0 \leq t \leq t_o, \quad |s| \leq t\}, \quad (t_o := \frac{2\delta}{\sqrt{4\delta^2 + \mu^2}}). \quad (48)$$

(a1). For the function \tilde{R} defined by (47), a direct computation shows that $d\tilde{R} \neq 0$ for (s, t) belonging to the interior of \tilde{D} defined by (48). So the maximal and minimal values of \tilde{R} are taken on the boundary of \tilde{D} .

(a2). On the boundary $|s| = t$, we have

$$\varphi(t) := \tilde{R}(\pm t, t) = \frac{(1-t^2)^3[(7\mu^2 + 40\delta^2)t^2 + \mu^2 - 8\delta^2]}{\mu(1+t^2)^4}, \quad 0 \leq t \leq t_o.$$

It is easy to see that $\varphi'(t) = 0$ has a unique solution $t_1 \in (0, t_o)$, where

$$t_1 = \frac{\sqrt{2}\delta}{\sqrt{\mu^2 + 6\delta^2}}.$$

(a3). On the boundary $t = t_o$, we have

$$\psi(s) := \tilde{R}(s, t_o) = \frac{\mu^5}{2(\mu^2 + 4\delta^2)^2} \frac{1 + 3s^2}{(1 + s^2)^3}, \quad s \in (-t_o, t_o).$$

It is easy to see that $\psi'(s) = 0$ has a unique solution s_1 , where $s_1 = 0$.

Summing up on (a1)–(a3), the maximal and minimal values of $\tilde{R}(s, t)$ on \tilde{D} are obtained from the following four values

$$\tilde{R}(0, 0), \quad \tilde{R}(\pm t_1, t_1), \quad \tilde{R}(\pm t_o, t_o), \quad \tilde{R}(s_1, t_o).$$

Therefore, the minimal and maximal values of the scalar flag curvature K_0^+ on S^n are given respectively by

$$\text{Min}(K_0^+) = \tilde{R}(0, 0) = \frac{\mu^2 - 8\delta^2}{\mu}, \quad \text{Max}(K_0^+) = \tilde{R}(\pm t_1, t_1) = \frac{(\mu^2 + 4\delta^2)^4}{\mu(\mu^2 + 8\delta^2)^3}$$

Then we obtain the proof of (14).

Case (C) : For $F = F_0^-$, first note that $\mu^2 > 12\delta^2$ in the following discussion (see Remark 5.1 below). In this case, by (46) we get

$$K_0^- = \frac{\mu^5 [12\mu(1 + s^2)^2 c^2 + (\mu^2 - 4\delta^2)(1 - s^2)(1 - 3s^2)]}{(\mu^2 - 4\delta^2 + 4\mu c^2)^3 (1 - s^2)^4}.$$

Then in Remark 4.2, $\tilde{R} = \tilde{R}(s, t)$ and \tilde{D} are given by

$$\tilde{R} = \frac{(1 + t^2)^2 [2(4\delta^2 - \mu^2)(1 + 5s^2)t^2 + 3\mu^2 s^4 - 4(\mu^2 - 10\delta^2)s^2 + \mu^2 + 8\delta^2]}{\mu(1 - s^2)^4}, \quad (49)$$

$$\tilde{D} = \{(s, t) | 0 \leq t \leq t_o, \quad |s| \leq t\}, \quad (t_o := \frac{2\delta}{\sqrt{\mu^2 - 4\delta^2}}). \quad (50)$$

(b1). For the function \tilde{R} defined by (49), a direct computation shows that $d\tilde{R} \neq 0$ for (s, t) belonging to the interior of \tilde{D} defined by (50). So the maximal and minimal values of \tilde{R} are taken on the boundary of \tilde{D} .

(b2). On the boundary $|s| = t$, we have

$$\lambda(t) := \tilde{R}(\pm t, t) = \frac{(1 + t^2)^3 [(40\delta^2 - 7\mu^2)t^2 + \mu^2 + 8\delta^2]}{\mu(1 - t^2)^4}, \quad 0 \leq t \leq t_o.$$

It is easy to see that $\lambda'(t) = 0$ has a unique solution $t_1 \in (0, t_o)$, where

$$t_1 = \frac{\sqrt{2} \delta}{\sqrt{\mu^2 - 6\delta^2}}.$$

(b3). On the boundary $t = t_o$, we have

$$\chi(s) := \tilde{R}(s, t_o) = \frac{\mu^5}{2(\mu^2 - 4\delta^2)^2} \frac{1 - 3s^2}{(1 - s^2)^3}, \quad s \in (-t_o, t_o).$$

It is easy to see that $\chi'(s) = 0$ has a unique solution $s_1 \in (-t_o, t_o)$, where $s_1 = 0$.

Summing up on (b1)–(b3), the maximal and minimal values of $\tilde{R}(s, t)$ on \tilde{D} are obtained from the following four values

$$\tilde{R}(0, 0), \quad \tilde{R}(\pm t_1, t_1), \quad \tilde{R}(\pm t_o, t_o), \quad \tilde{R}(s_1, t_o).$$

Therefore, the minimal and maximal values of the scalar flag curvature K_0^- on S^n are given respectively by

$$\text{Min}(K_0^-) = \tilde{R}(\pm t_o, t_o) = \frac{\mu^5(\mu^2 - 16\delta^2)}{(\mu^2 - 8\delta^2)^3}, \quad \text{Max}(K_0^-) = \tilde{R}(\pm t_1, t_1) = \frac{(\mu^2 - 4\delta^2)^4}{\mu(\mu^2 - 8\delta^2)^3}.$$

Then we obtain the proof of (15).

Q.E.D.

Remark 5.1 In Theorem 1.2, it is easily seen from (44) that $F = F_2^+$ and $F = F_0^+$ are regular if and only if the constants μ, δ satisfy $\mu > 0$ and $\delta \geq 0$; $F = F_0^-$ is regular if and only if the constants μ, δ satisfy $\mu^2 > 12\delta^2$. The former follows from (44) and $b^2 < 1$, and the latter holds since by (44) we get $b^2 < 1/2 \Leftrightarrow c^2 > (12\delta^2 - \mu^2)/(12\mu)$, and thus we have $12\delta^2 - \mu^2 < 0$ from $\text{Min}_{x \in S^n} c(x) = 0$.

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